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THE GROWTH OF A THERMAL LAYER IN A POROUS MEDIUM ADJACENT TO A SUDDENLY HEATED SEMI-INFINITE HORIZONTAL SURFACE

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NOMENCLATURE

A	approximating function defined in equation (24)
a	approximating function defined in equation (23)
C_p	specific heat at constant pressure
f	weighting function in equation (21)
g	gravitational acceleration
K	permeability of the porous medium
k	thermal conductivity of the saturated porous medium
L	characteristic length of the plate
Nu_x	local Nusselt number, $q_w x/k(T_w - T_\infty)$
q_w	surface heat flux
Ra	Rayleigh number based on the characteristic length L, $K\rho_\infty g\beta(T_w - T_\infty)L/\mu\alpha$
Ra_x	local Rayleigh number $K\rho_\infty g\beta(T_w - T_\infty)x/\mu\alpha$
T	temperature
t	time
U	dimensionless Darcian velocity in the x-direction, $(L/\alpha Ra^{2/3})u$
u	Darcian velocity in the x-direction
V	dimensionless Darcian velocity in the y-direction, $(L/\alpha Ra^{1/3})v$
v	Darcian velocity in the y-direction
X	dimensionless distance in the x-direction, x/L
x	coordinate along the horizontal plate
Y	dimensionless distance in the y-direction, $yRa^{1/3}/L$
y	coordinate perpendicular to the plate pointing toward the porous medium

Greek symbols

α	equivalent thermal diffusivity
β	coefficient of thermal expansion
σ	ratio of heat capacity of the saturated porous medium to that of the fluid
Δ	dimensionless boundary layer thickness, δ/L
δ	boundary layer thickness
ε	porosity
ζ	variable defined in equation (9), Y/Δ
η	variable defined in equation (17)
θ	dimensionless temperature, $(T - T_\infty)/(T_w - T_\infty)$
μ	viscosity of fluid
ρ	density of the fluid
τ	dimensionless time, $\alpha t Ra^{2/3}/\sigma L$

Superscript

*	quantities associated with the case of constant heat flux

Subscript

∞	condition at infinity
w	condition at the wall

1. INTRODUCTION

When the wall temperature of a semi-infinite horizontal upward facing plate is suddenly raised to T_w which is higher than the ambient temperature of the fluid-filled porous medium at T_∞, a thermal boundary layer begins to grow above the heated plate. The growth of the thermal boundary layer is the subject of investigation in the present paper. By assuming a temperature profile that satisfies the boundary conditions, the governing equations are solved by the Karman–Pohlhausen integral method. The resulting nonlinear partial differential equation for the boundary layer thickness is of diffusion type that can be solved asymptotically for small and large times. For small time when the leading edge effect is not being felt, heat is transferred by transient 1-dim. heat conduction. For large time, solutions obtained in this paper agree well with the exact similarity solution for steady free convection obtained in a previous study [1]. Approximate solutions valid for all times for the growth of the boundary layer thickness are also obtained based on the method of integral relations which has been shown to be accurate for engineering applications [2].

2. FORMULATION

The dimensionless boundary layer equations for the problem of transient free convection in a porous medium adjacent to a semi-infinite isothermal, heated horizontal surface located along the positive x-axis are

$$\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} = 0, \tag{1}$$

$$\frac{\partial U}{\partial Y} = \frac{\partial \theta}{\partial X}, \tag{2}$$

$$\frac{\partial \theta}{\partial \tau} + U\frac{\partial \theta}{\partial X} + V\frac{\partial \theta}{\partial Y} = \frac{\partial^2 \theta}{\partial Y^2}. \tag{3}$$

The dimensionless quantities in equations (1)–(3) are related to their corresponding dimensional variables through the following definitions:

$$\tau = \alpha t Ra^{2/3}/\sigma L, \quad X = x/L, \quad Y = yRa^{1/3}/L,$$
$$U = (L/\alpha Ra^{2/3})u, \quad V = (L/\alpha Ra^{1/3})v, \tag{4}$$
$$\theta = (T - T_\infty)/(T_w - T_\infty)$$

where u and v are the Darcian velocities in the x- and y-directions. T_w and T_∞ are the temperatures at the wall and at infinity. L is a characteristic length.

$$Ra = \rho_\infty g\beta K(T_w - T_\infty)L/\mu\alpha$$

is the modified Rayleigh number with ρ_∞, β and μ being the density of the fluid at infinity, the thermal expansion coefficient and the viscosity of the fluid. K is the intrinsic permeability of the porous medium. g is the gravitational acceleration. $\alpha = k/(\rho C_p)_f$ is the equivalent thermal diffusivity with k denoting the stagnant thermal conductivity of the saturated porous medium and $(\rho C_p)_f$ the heat capacity of the fluid.

$$\sigma = [\varepsilon(\rho C_p)_f + (1-\varepsilon)(\rho C_p)_m]/(\rho C_p)_f$$

is the ratio of the heat capacity of the saturated porous medium to that of the fluid with ε denoting the porosity.

The initial and boundary conditions for the problem under consideration are

$$U(X,Y,0) = V(X,Y,0) = \theta(X,Y,0) = 0, \tag{5}$$

$$V(X,0,\tau) = 0, \quad \theta(X,0,\tau) = 1, \tag{6a, b}$$

$$U(X,\infty,\tau) = \theta(X,\infty,\tau) = 0. \tag{7a, b}$$

We now attempt to solve equations (1)–(3) subject to the initial and boundary conditions (5)–(7) approximately by the Karman–Pohlhausen integral method. To this end, we first rewrite equation (3) in divergent form and integrate the resulting equation with the aid of equation (1) to give

$$\frac{\partial}{\partial \tau}\int_0^\infty \theta \, \mathrm{d}Y + \frac{\partial}{\partial X}\int_0^\infty U\theta \, \mathrm{d}Y = -\left(\frac{\partial \theta}{\partial Y}\right)_{Y=0} \tag{8}$$

Next, we assume that the temperature distribution is of the form

$$\theta = \mathrm{erfc}(\zeta) \tag{9}$$

where $\zeta = Y/\Delta(X,\tau)$ with Δ denoting the dimensionless boundary layer thickness and $\mathrm{erfc}(\zeta)$ the complementary error function. Substituting equation (9) into equation (2) and integrating, one obtains

$$U = \frac{1}{\sqrt{\pi}} \mathrm{e}^{-\zeta^2} \frac{\partial \Delta}{\partial X} \tag{10}$$

where the boundary condition (7a) has been imposed. Substitution of equations (9) and (10) in equation (8) and after integration yields

$$\frac{\partial \Delta}{\partial \tau} + \frac{\sqrt{\pi}}{4}\frac{\partial}{\partial X}\left(\Delta \frac{\partial \Delta}{\partial X}\right) = \frac{2}{\Delta} \tag{11}$$

which is the governing equation for the growth of the thermal boundary layer in a porous medium adjacent to a semi-infinite horizontal heated plate. Equation (11) is a second-order partial differential equation of diffusion type as opposed to the first-order hyperbolic equation for transient Darcian free convection about a vertical plate [2].

3. ASYMPTOTIC SOLUTIONS

We now obtain the asymptotic solutions of equation (11) for small and large times.

(i) For small time, we have $\Delta = \Delta(\tau)$. Equation (11) with the second term neglected and subject to the initial condition $\Delta(0) = 0$ gives

$$\Delta(\tau) = 2\tau^{1/2}. \tag{12}$$

Substitution of equation (12) in equations (9) and (10) yields

$$\theta = \mathrm{erfc}\left(\frac{Y}{2\tau^{1/2}}\right), \tag{13}$$

$$U = 0. \tag{14}$$

Equations (13) and (14) show that during the initial stage, the fluid is motionless and heat is transferred by transient 1-dim. heat conduction. The small time solution ceases to be valid when convective motion begins.

(ii) For large time, i.e. at steady state, we have $\Delta = \Delta(X)$. Equation (11) with the transient term neglected reduces to

$$\Delta(\Delta\Delta')' = \frac{8}{\sqrt{\pi}} \tag{15}$$

where the primes denote the derivatives with respect to X. Solution of equation (15) subject to the boundary condition $\Delta = 0$ at $X = 0$ is

$$\Delta(X) = \left(\frac{36}{\sqrt{\pi}}\right)^{1/3} X^{2/3} = 2.728X^{2/3}. \tag{16}$$

Substitution of equation (16) into equation (9) yields

$$\theta = \mathrm{erfc}\left(\frac{\sqrt{\pi}}{36}\right)^{1/3} \eta \tag{17}$$

where $\eta = Ra_x^{1/3} Y/X$ is the similarity variable of the steady problem [1]. It follows from equations (10) and (16) that the dimensionless horizontal velocity profile is

$$UX^{1/3} = \frac{2}{3\sqrt{\pi}}\left(\frac{36}{\sqrt{\pi}}\right)^{1/3} \exp\left[-\left(\frac{\sqrt{\pi}}{36}\right)^{2/3}\eta^2\right]. \tag{18}$$

The local Nusselt number, Nu_x, is defined as

$$Nu_x = \frac{xq_w}{k(T_w - T_\infty)} \tag{19}$$

where

$$q_w = -k\left(\frac{\partial T}{\partial y}\right)_{y=0}$$

is the local surface heat flux. Substitution of equation (17) into equation (19) yields

$$\frac{Nu_x}{Ra_x^{1/3}} = \frac{2}{\sqrt{\pi}}\left(\frac{\sqrt{\pi}}{36}\right)^{1/3} = 0.4135 \tag{20}$$

which agrees very well with the exact value of 0.4200 obtained from the similarity solution [4].

4. APPROXIMATE SOLUTIONS VALID FOR ALL TIMES

We now obtain an approximate solution for equation (11) based on the method of integral relations [2, 3]. To this end, we multiply equation (11) by a weighting function $f(X)$ with the resulting equation integrated from $X = 0$ to $X = 1$ to give

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\int_0^1 f\Delta \, \mathrm{d}x + \frac{\sqrt{\pi}}{4}\left(f\Delta\frac{\partial \Delta}{\partial X}\right)_{x=1} - \frac{\sqrt{\pi}}{4}\int_0^1 f'\Delta\frac{\partial \Delta}{\partial X}\,\mathrm{d}X = 2\int_0^1 \frac{f}{\Delta}\,\mathrm{d}X. \tag{21}$$

Next, we assume that

$$f(X) = X \tag{22}$$

and

$$\Delta(X,\tau) = a(\tau)X^{2/3} \tag{23}$$

where the weighting function given by equation (22) has been used by Heinisch *et al.* [3] and by Cheng and Pop [2]. Substituting equations (22) and (23) into equation (21) results in the following ordinary differential equation:

$$\frac{\mathrm{d}A}{\mathrm{d}\tau} + \frac{2\sqrt{\pi}}{9}A^{3/8} = 8 \tag{24}$$

where $A = a^2$. Equation (24) with the initial condition $A = 0$ at $\tau = 0$ is integrated numerically by Hamming's method [5] with a step size of $\Delta\tau = 10^{-4}$.

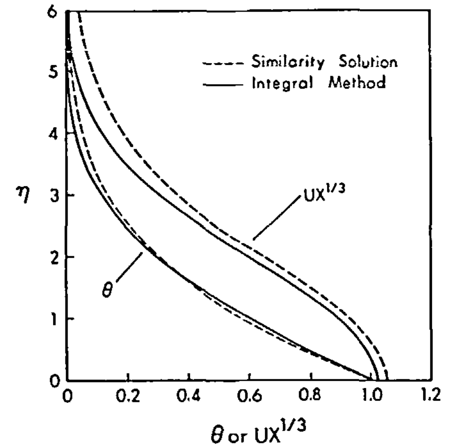

FIG. 1. Comparison of approximate and exact steady state dimensionless temperature and horizontal velocity profiles.

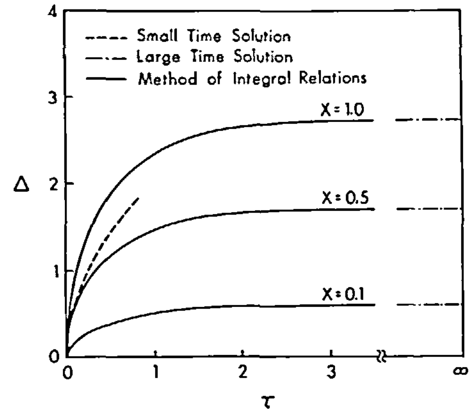

FIG. 2. The growth of the dimensionless thermal boundary layer thickness versus dimensionless time.

5. COMPARISON OF RESULTS

Figure 1 is a plot of the dimensionless temperature and horizontal velocity profiles for steady state Darcian free convection about a horizontal heated plate as obtained by the similarity solution and the approximate solution for large time given by equations (17) and (18). As shown in the figure, the approximate solutions for large time agree well with the exact similarity solution.

The growth of the dimensionless boundary layer thickness as a function of dimensionless time at $X = 0.1, 0.5$ and 1.0 is shown in Fig. 2, where the solid lines are the approximate solution given by equation (23) with the function $a(\tau)$ obtained from the numerical solution of equation (24). The asymptotic solutions given by equations (12) and (16) for small and large times are plotted as dashed lines in the same graph for comparison.

REFERENCES

1. P. Cheng and I.-D. Chang, Buoyancy induced flows in a saturated porous medium adjacent to impermeable horizontal surfaces, *Int. J. Heat Mass Transfer* **19**, 1267–1272 (1976).
2. P. Cheng and I. Pop, Transient free convection about a vertical flat plate embedded in a porous medium, *Int. J. Engng Sci.* (1983) in press.
3. R. P. Heinisch, R. Viskanta and R. M. Singer, Approximate solution of transient free convection laminar boundary layer equations, *Z. Angew. Math. Phys.* **20**, 19–33 (1969).
4. P. Cheng, Convective heat transfer in a porous layers by integral methods, *Lett. Heat Mass Transfer* **5**, 243–252 (1978).
5. R. W. Hornbeck, *Numerical Methods*, p. 200. Quantum, New York (1975).